FB 10 - Institut für Mathematik
Algorithmische Algebra und
Diskrete Mathematik
Prof. i.R. Dr. Wolfram Koepf

OBERSEMINAR
Algorithmische Algebra und Diskrete Mathematik

The Tutte Polynomial of Ideal Arrangements

Referent:	**Hery Randriamaro** **(Gast aus Madagaskar, Stipendiat der AvH-Stiftung,** **Institut für Mathematik der Universität Kassel)**
Termin:	Mittwoch, 28. Juli 2021, 15.15 Uhr
Ort:	Raum 2420, Heinrich-Plett-Str. 40, AVZ, Kassel-Oberzwehren

Abstract:

The Tutte polynomial was at the origin in 1954 a bivariate polynomial enumerating the colorings of a graph and of its dual graph. But it reveals more of the internal structure of the graph like its number of forests, of spanning subgraphs, and of acyclic orientations.
Ardila extended in 2007 the notion of Tutte polynomial to hyperplane arrangements. At the same time, he computed the Tutte polynomial of the hyperplane arrangements associated with the root systems of the classical Weyl groups. The Tutte polynomials associated to the root systems of the exceptional Weyl groups were computed by De Concini and Procesi one year later.
We consider the hyperplane arrangements associated with ideals of the root system of a Weyl group. These arrangements were introduced in 2006 by Sommers and Tymoczko. The talk assumes that the subject does not belong to the main field of expertise of a significant part of the audience. That is why enough time will be taken to define important notions like the Tutte polynomial, the Weyl groups, and the ideals of a root system even if they could seem to be basic.
We also expose our results from 2020 concerning the Tutte polynomial of hyperplane arrangements associated with ideals of a classical root system. Then, we finish with an introduction of an open problem on the Tutte polynomial of hyperplane arrangements associated with ideals of an exceptional root system.

gez. Prof. Dr. Wolfram Koepf

The Tutte Polynomial of Ideal Arrangements

Hery Randriamaro

Universität Kassel

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Tutte Polynomial

We work on the Euclidean space $\mathbb{R}^n$.

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Take a vector $a = (a_1, \dots, a_n)$, and real variables $x_1, \dots, x_n$. A **hyperplane** is a subspace $a^\perp := \{a_1x_1 + \cdots + a_nx_n = 0\}$ of $\mathbb{R}^n$. A **hyperplane arrangement** is a finite set of hyperplanes.

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The **rank** of a hyperplane arrangement $\mathcal{A}$ is $\mathrm{rk}\,\mathcal{A} := n - \dim \bigcap_{H \in \mathcal{A}} H$.

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The **rank** of a hyperplane arrangement $\mathcal{A}$ is $\mathrm{rk}\,\mathcal{A} := n - \dim \bigcap_{H \in \mathcal{A}} H$.

The **Tutte polynomial** of a hyperplane arrangement $\mathcal{A}$ is

$$T_{\mathcal{A}}(x, y) := \sum_{\mathcal{B} \subseteq \mathcal{A}} (x-1)^{\mathrm{rk}\,\mathcal{A} - \mathrm{rk}\,\mathcal{B}} (y-1)^{\#B - \mathrm{rk}\,\mathcal{B}}.$$

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Tutte Polynomial

The complexification of the hyperplane $H = \{a_1x_1 + \cdots + a_nx_n = 0\}$ is the complex hyperplane $H_{\mathbb{C}} := \{a_1z_1 + \cdots + a_nz_n = 0\}$ with complex variables $z_1, \ldots, z_n$.

Let $M_{\mathcal{A}} := \mathbb{R}^n \setminus \bigcup_{H \in \mathcal{A}} H$ and $M_{\mathcal{A}_{\mathbb{C}}} := \mathbb{C}^n \setminus \bigcup_{H \in \mathcal{A}} H_{\mathbb{C}}$:

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- the number of connected component of $M_{\mathcal{A}}$ is $T_{\mathcal{A}}(2, 0)$,

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- the number of connected component of $M_{\mathcal{A}}$ is $T_{\mathcal{A}}(2, 0)$,
- the number of bounded connected component of $M_{\mathcal{A}}$ is $(-1)^{\mathrm{rk}\,\mathcal{A}} T_{\mathcal{A}}(0, 0)$,

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The complexification of the hyperplane $H = \{a_1x_1 + \cdots + a_nx_n = 0\}$ is the complex hyperplane $H_{\mathbb{C}} := \{a_1z_1 + \cdots + a_nz_n = 0\}$ with complex variables $z_1, \ldots, z_n$.

Let $M_{\mathcal{A}} := \mathbb{R}^n \setminus \bigcup_{H\in\mathcal{A}} H$ and $M_{\mathcal{A}_{\mathbb{C}}} := \mathbb{C}^n \setminus \bigcup_{H\in\mathcal{A}} H_{\mathbb{C}}$:

- the number of connected component of $M_{\mathcal{A}}$ is $T_{\mathcal{A}}(2, 0)$,
- the number of bounded connected component of $M_{\mathcal{A}}$ is $(-1)^{\mathrm{rk}\,\mathcal{A}} T_{\mathcal{A}}(0, 0)$,
- the Poincaré polynomial of the cohomology ring of $M_{\mathcal{A}_{\mathbb{C}}}$ is

$$\sum_{k\in\mathbb{N}} \mathrm{rank}\, H^k(M_{\mathcal{A}_{\mathbb{C}}}, \mathbb{Z})\, q^k = (-1)^{\mathrm{rk}\,\mathcal{A}} q^{n-\mathrm{rk}\,\mathcal{A}} T_{\mathcal{A}}(1 - q, 0).$$

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Root System

The **reflection** associated to the hyperplane $a^\perp$ is the orthogonal transformation $s_{a^\perp} : \mathbb{R}^n \to \mathbb{R}^n$ defined by $s_{a^\perp}(x) := x - 2\dfrac{\langle x, a\rangle}{\langle a, a\rangle}a$.

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A hyperplane arrangement $\mathcal{A}$ is a **Coxeter arrangement** if

$$\forall H, H' \in \mathcal{A} :\ s_H(H') \in \mathcal{A}.$$

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Let $\mathcal{A}$ be a Coxeter arrangement, and Φ a set of vectors a such that $a^\perp \in \mathcal{A}$. Then Φ is a **root system** of $\mathcal{A}$ if

- $\forall a \in \Phi : \; \Phi \cap \mathbb{R}a = \{a, -a\}$,
- $\forall a \in \Phi : \; s_{a^\perp}(\Phi) = \Phi$.

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A vector of a root system is called a **root**.

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Crystallographic Root System

A root system Φ is **crystallographic** if for every $u, v \in \Phi$, $2\dfrac{\langle u, v\rangle}{\langle v, v\rangle} \in \mathbb{Z}$.

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Crystallographic Root System

A root system Φ is **crystallographic** if for every $u, v \in \Phi$, $2\dfrac{\langle u, v\rangle}{\langle v, v\rangle} \in \mathbb{Z}$.

A root system Φ is **reducible** if there are two nonempty subsets $\Phi_1, \Phi_2 \subseteq \Phi$ such that $\Phi = \Phi_1 \sqcup \Phi_2$ and $\langle u_1, u_2\rangle = 0$ for every $(u_1, u_2) \in \Phi_1 \times \Phi_2$.

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Crystallographic Root System

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$$\begin{aligned}
(n \geq 2) \quad & \Phi_{A_{n-1}} = \{e_i - e_j \mid 1 \leq i \neq j \leq n\},\\
(n \geq 2) \quad & \Phi_{B_n} = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{\pm e_i \mid i \in [n]\},\\
(n \geq 2) \quad & \Phi_{C_n} = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{\pm 2e_i \mid i \in [n]\},\\
(n \geq 4) \quad & \Phi_{D_n} = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\},
\end{aligned}$$

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Crystallographic Root System

and the five exceptional root systems

$$\Phi_{G_2} = \big\{ \pm (e_i - e_j) \big\}_{1 \leq i < j \leq 3} \sqcup \big\{ \pm (2e_i - e_j - e_k) \big\}_{\{i,j,k\} = \{1,2,3\}},$$

$$\Phi_{F_4} = \{\pm e_i\}_{i \in [4]} \sqcup \{\pm e_i \pm e_j\}_{1 \leq i < j \leq 4} \sqcup \Big\{ \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4) \Big\},$$

$$\Phi_{E_8} = \{\pm e_i \pm e_j\}_{1 \leq i < j \leq 8} \sqcup \Big\{ \frac{1}{2} \sum_{i=1}^{8} \pm e_i \text{ even number of } + \Big\},$$

$$\Phi_{E_7} = \{\pm e_i \pm e_j\}_{1 \leq i < j \leq 6} \sqcup \big\{ \pm (e_7 - e_8) \big\}$$

$$\sqcup \Big\{ \pm \frac{1}{2}\big(e_7 - e_8 + (\sum_{i=1}^{6} \pm e_i \text{ odd number of } +)\big) \Big\},$$

$$\Phi_{E_6} = \{\pm e_i \pm e_j\}_{1 \leq i < j \leq 5} \sqcup \Big\{ \pm \frac{1}{2}\big(e_8 - e_7 - e_6 + (\sum_{i=1}^{5} \pm e_i \text{ odd number of } +)\big) \Big\}.$$

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Root System Ideal

Let Φ be a crystallographic root system.

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Root System Ideal

Let Φ be a crystallographic root system. There exist some subsets $\Delta \subseteq \Phi$, called **simple systems**, such that $\langle \Delta \rangle = \mathbb{R}^n$ and each root in Φ is a linear combination of roots in Δ with coefficients all of the same sign.

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Endow Φ^+ with the partial order $\preceq$ defined for all $u, v \in \Phi^+$ by

$$u \preceq v \Longleftrightarrow v - u \in \mathbb{N}\Phi^+.$$

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Root System Ideal

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Endow Φ^+ with the partial order $\preceq$ defined for all $u, v \in \Phi^+$ by

$$u \preceq v \Longleftrightarrow v - u \in \mathbb{N}\Phi^+.$$

An **ideal** of a crystallographic root system Φ is a subset $I \subseteq \Phi^+$ such that

$$\text{If } u \in I, \text{ and } v \in \Phi^+ \text{ so that } u \preceq v, \text{ then } v \in I.$$

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Ideal Arrangement

The **ideal arrangement** $\mathcal{A}_I$ associated to an ideal I of a crystallographic root system Φ is the hyperplane arrangement defined by

$$\mathcal{A}_I := \{u^\perp \mid u \in \Phi^+ \setminus I\}.$$

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Sommers and Tymoczko showed in 2006 that the arrangement $\mathcal{A}_\mathcal{I}$ is free if I is an ideal of a classical root system or of Φ_{G_2}.

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Coboundary Polynomial

The **coboundary polynomial** of a hyperplane arrangement $\mathcal{A}$ is

$$\bar{\chi}_{\mathcal{A}}(q,t) := \sum_{\mathcal{B}\subseteq\mathcal{A}} q^{\operatorname{rk}\mathcal{A}-\operatorname{rk}\mathcal{B}}(t-1)^{\#\mathcal{B}}.$$

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$$\bar{\chi}_{\mathcal{A}}(q, t) := \sum_{\mathcal{B} \subseteq \mathcal{A}} q^{\operatorname{rk} \mathcal{A} - \operatorname{rk} \mathcal{B}} (t-1)^{\# \mathcal{B}}.$$

Since $T_{\mathcal{A}}(x, y) = \dfrac{\bar{\chi}_{\mathcal{A}}\big((x-1)(y-1), y\big)}{(y-1)^{\operatorname{rk} \mathcal{A}}}$, computing the coboundary polynomial is equivalent to computing the Tutte polynomial.

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Correct Reduction

Define the semilattice $L(\mathcal{A}) := \Big\{ \bigcap_{H \in \mathcal{B}} H \;\Big|\; \mathcal{B} \subseteq \mathcal{A} \Big\}$ partially ordered by reverse inclusion.

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Correct Reduction

Define the semilattice $L(\mathcal{A}) := \left\{ \bigcap_{H \in \mathcal{B}} H \;\middle|\; \mathcal{B} \subseteq \mathcal{A} \right\}$ partially ordered by reverse inclusion. Two hyperplane arrangements $\mathcal{A}$ and $\mathcal{B}$ are **isomorphic** if there is an order preserving bijection between the $L(\mathcal{A})$ and $L(\mathcal{B})$.

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Correct Reduction

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For a $\mathbb{Z}$-hyperplane $H = \{a_1 x_1 + \dots + a_n x_n = 0\}$ in $\mathbb{R}^n$ and a prime number p, define the set $\bar{H} = \{\bar{a}_1 \bar{x}_1 + \dots + \bar{a}_n \bar{x}_n = \bar{0}\}$ in $\mathbb{F}_p^n$.

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Correct Reduction

Define the semilattice $L(\mathcal{A}) := \left\{ \bigcap_{H \in \mathcal{B}} H \;\middle|\; \mathcal{B} \subseteq \mathcal{A} \right\}$ partially ordered by reverse inclusion. Two hyperplane arrangements $\mathcal{A}$ and $\mathcal{B}$ are **isomorphic** if there is an order preserving bijection between the $L(\mathcal{A})$ and $L(\mathcal{B})$.

For a $\mathbb{Z}$-hyperplane $H = \{a_1 x_1 + \cdots + a_n x_n = 0\}$ in $\mathbb{R}^n$ and a prime number p, define the set $\bar{H} = \{\bar{a}_1 \bar{x}_1 + \cdots + \bar{a}_n \bar{x}_n = \bar{0}\}$ in $\mathbb{F}_p^n$. One says that a $\mathbb{Z}$-arrangement $\mathcal{A}$ **reduces correctly** over $\mathbb{F}_p$ if

- for every hyperplane H in $\mathcal{A}$, $\bar{H}$ is a hyperplane in $\mathbb{F}_p^n$,
- and, if we define $\bar{\mathcal{A}} := \{\bar{H} \mid H \in \mathcal{A}\}$, $\mathcal{A}$ and $\bar{\mathcal{A}}$ are isomorphic.

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Finite Field Method

For a hyperplane arrangement $\bar{\mathcal{A}}$ and a vector $\bar{x}$ in $\mathbb{F}_p^n$, define the arrangement

$$\bar{\mathcal{A}}(\bar{x}) := \{\bar{H} \in \bar{\mathcal{A}} \mid \bar{x} \in \bar{H}\}.$$

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For a hyperplane arrangement $\bar{\mathcal{A}}$ and a vector $\bar{x}$ in $\mathbb{F}_p^n$, define the arrangement

$$\bar{\mathcal{A}}(\bar{x}) := \{\bar{H} \in \bar{\mathcal{A}} \mid \bar{x} \in \bar{H}\}.$$

Consider a $\mathbb{Z}$-arrangement $\mathcal{A}$ in $\mathbb{R}^n$ that reduces correctly over $\mathbb{F}_p$. Then

$$p^{n-\mathrm{rk}\,\mathcal{A}}\bar{\chi}_{\mathcal{A}}(p,t) = \sum_{\bar{x}\in\mathbb{F}_p^n} t^{\#\bar{\mathcal{A}}(\bar{x})}.$$

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Example

Consider the arrangement $\mathcal{A}_{A_{n-1}} = \big\{\{x_i - x_j = 0\}\big\}_{1 \leq i < j \leq n}$.

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Example

Consider the arrangement $\mathcal{A}_{A_{n-1}} = \big\{\{x_i - x_j = 0\}\big\}_{1 \leq i < j \leq n}$.

We have $\operatorname{rk} \mathcal{A}_{A_{n-1}} = n - 1$ and

$$\bar{\mathcal{A}}_{A_{n-1}}(\bar{x}) = \binom{\#X_0}{2} + \cdots + \binom{\#X_{p-1}}{2} \text{ with } X_k = \{i \in [n] \mid \bar{x}_i = k\}.$$

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Thus $q\bar{\chi}_{\mathcal{A}_{A_{n-1}}}(p, t) = \displaystyle\sum_{X_0 \sqcup \cdots \sqcup X_{p-1} = [n]} t^{\binom{\#X_0}{2} + \cdots + \binom{\#X_{p-1}}{2}}$.

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Thus $$q\bar{\chi}_{\mathcal{A}_{A_{n-1}}}(p, t) = \sum_{X_0 \sqcup \cdots \sqcup X_{p-1} = [n]} t^{\binom{\#X_0}{2} + \cdots + \binom{\#X_{p-1}}{2}}.$$

And $$1 + q \sum_{n \in \mathbb{N}^*} \bar{\chi}_{\mathcal{A}_{A_{n-1}}}(p, t) \frac{a^n}{n!} = \left(\sum_{n \in \mathbb{N}} t^{\binom{n}{2}} \frac{a^n}{n!} \right)^p.$$

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Tutte Polynomial of Classical Ideal

Let $\mathcal{A}_I$ be an ideal arrangement of $\Phi_{A_{n-1}}$ with partition $A^{(1)}|\dots|A^{(r)}$, and let $R^{(u)} = \big\{v \in \{u+1,\dots,r\} \;\big|\; s_I(A^{(u)}) \cap s_I(A^{(v)}) \neq \emptyset\big\}$. Then, the coboundary polynomial of $\mathcal{A}_I$ is

$$\bar{\chi}_{\mathcal{A}_I}(p,t) = \sum_{\substack{a_1^{(1)}+\dots+a_p^{(1)} \,=\, \#A^{(1)} \\ \vdots \\ a_1^{(r)}+\dots+a_p^{(r)} \,=\, \#A^{(r)}}} \prod_{u=1}^{r} \binom{\#A^{(u)}}{a_1^{(u)},\dots,a_p^{(u)}} \frac{t^{\sum_{s=1}^{p} \binom{a_s^{(u)}}{2} + a_s^{(u)} \sum_{v \in R^{(u)}} a_s^{(v)}}}{p}.$$

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Active Elements

Let X be a vector set in $\mathbb{R}^n$.

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Let X be a vector set in $\mathbb{R}^n$. The **rank** of $A \subseteq X$ is $\operatorname{rk} A := \operatorname{rk} \{a^{\perp}\}_{a \in A}$.

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Define the set $A_{\lhd x} := \{a \in A \mid a \lhd x\}$.

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Take a basis B in $\mathscr{B}(X)$:

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Take a basis B in $\mathscr{B}(X)$:

- Let $b \in B$. One says that b is an **internal active** element of B if

$$\forall x \in X_{\lhd b} \setminus B : \ \operatorname{rk}\big(\{x\} \sqcup (B \setminus \{b\})\big) < n.$$

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- Let $x \in X \setminus B$. One says that x is an **external active** element of B if

$$\operatorname{rk}\big(\{x\} \sqcup B_{\rhd x}\big) = \operatorname{rk}(B_{\rhd x}).$$

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Theorem of Crapo

Denote by $i(B)$ resp. $e(B)$ the number of internal resp. external active elements of a basis B.

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Theorem of Crapo

Denote by $i(B)$ resp. $e(B)$ the number of internal resp. external active elements of a basis B. The Tutte polynomial of a hyperplane arrangement $\mathcal{A} = \{x^\perp\}_{x \in X}$ is

$$T_{\mathcal{A}}(x, y) = \sum_{B \in \mathscr{B}(X)} x^{i(B)} y^{e(B)}.$$

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The algorithm from the definition of the Tutte polynomial would implement $\binom{\#X}{k}$ sets of cardinality k, where k varies from 1 to $\#X$.

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The formula of Crapo reduces the algorithm implementation on $\binom{\#X}{\mathrm{rk}\, X}$ sets of cardinality $\mathrm{rk}\, X$.

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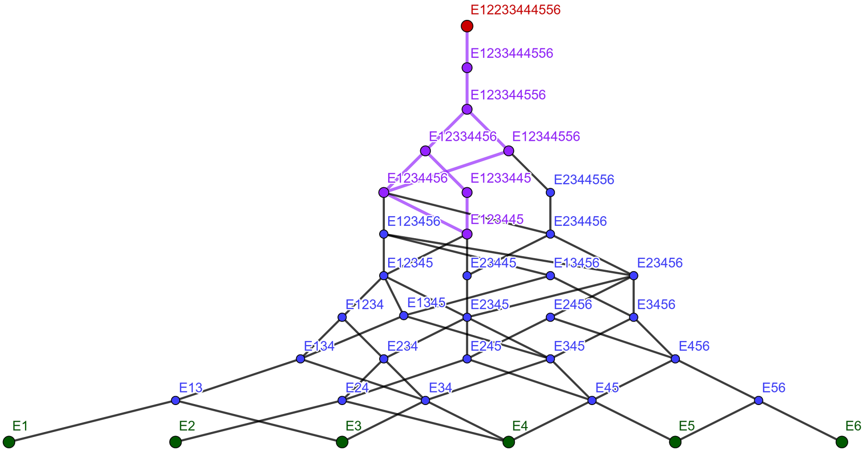
Graph Representation of E_6



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Example

$I = \big((1,1,1,2,1,0),(1,1,1,2,1,1),(1,1,2,2,1,0),(1,1,2,2,1,1),$
$(1,1,1,2,2,1),(1,1,2,2,2,1),(1,1,2,3,2,1),(1,2,2,3,2,1)\big)$ is an ideal of $\Phi_{E_6}^+$, and the Tutte polynomial of its associated ideal arrangement is

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Example

$I = \big((1,1,1,2,1,0),(1,1,1,2,1,1),(1,1,2,2,1,0),(1,1,2,2,1,1),$
$(1,1,1,2,2,1),(1,1,2,2,2,1),(1,1,2,3,2,1),(1,2,2,3,2,1)\big)$ is an ideal of $\Phi^+_{E_6}$, and the Tutte polynomial of its associated ideal arrangement is

$T_{\mathcal{A}_{I_e}}(x,y) = y^{22} + 6y^{21} + 21y^{20} + 56y^{19} + 126y^{18} + 252y^{17} + xy^{15}$
$+462y^{16} + 5xy^{14} + 791y^{15} + 18xy^{13} + 1281y^{14} + 52xy^{12} + 1978y^{13}$
$+129xy^{11} + 2927y^{12} + 295xy^{10} + 4163y^{11} + 5x^2y^8 + 623xy^9 + 5688y^{10} +$
$26x^2y^7 + 1212xy^8 + 7445y^9 + 110x^2y^6 + 2176xy^7 + 9288y^8 + 346x^2y^5 +$
$3596xy^6 + 10957y^7 + x^6 + 79x^3y^3 + 892x^2y^4 + 5404xy^5 + 12065y^6 +$
$22x^5 + 62x^4y + 303x^3y^2 + 1829x^2y^3 + 7235xy^4 + 12159y^5 + 191x^4$
$+762x^3y + 2863x^2y^2 + 8292xy^3 + 10860y^4 + 818x^3 + 3184x^2y +$
$7646xy^2 + 8136y^3 + 1728x^2 + 4872xy + 4584y^2 + 1440x + 1440y.$

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